Lecture 1: Introduction to Nonlinear Systems

Linear System

$$\dot{x} = \overset{n \times n \text{ constant matrix}}{A} x$$

$\dot{x} = \frac{d}{dt} x(t)$

state $\in \mathbb{R}^n$

$x(t_0) = x_0 \in \mathbb{R}^n$ ← initial condition

Properties of Linear system: (w/o ctrl input)

1. Solutions always exist

   → given in the closed form:

   $$x(t) = \exp\{A(t-t_0)\}x_0$$

   $$t \geq t_0$$

2. Solutions exist $\forall t \quad -\infty < t < \infty$

3. Solutions are unique

4. The equilibrium points x^* are the nullspace of A

5. Periodic solutions are only marginally stable or never stable

Nonlinear Systems (w/o Control input)

$\dot{x} = \underbrace{f(x)}_{\text{system dynamics}} \qquad x(t_0) \in \mathbb{R}^n$

$f: \mathbb{R}^n \to \mathbb{R}^n$

types

I. time-invariant (autonomous) system:

$$\dot{x} = f(x) \qquad f: \mathbb{R}^n \to \mathbb{R}^n$$

II. time-varying (non-autonomous) system:

$$\dot{x} = f(t, x) \qquad f: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$$

Linear (w/ control input) u

$$\dot{x} = Ax + Bu \longrightarrow \dot{x} = f(x, u)$$

Control Affine Form $\quad \dot{x} = f(x) + g(x)u$

Control Analysis : Determine stability?

Convergence?

$\dot{x} = F(x)$

Control Design: choose u as a fcn of x

$\dot{x} = F(x) + g(x)\,u$

used to model population dynamics

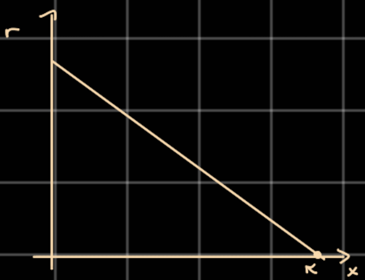
Logistic Growth Model (Example)

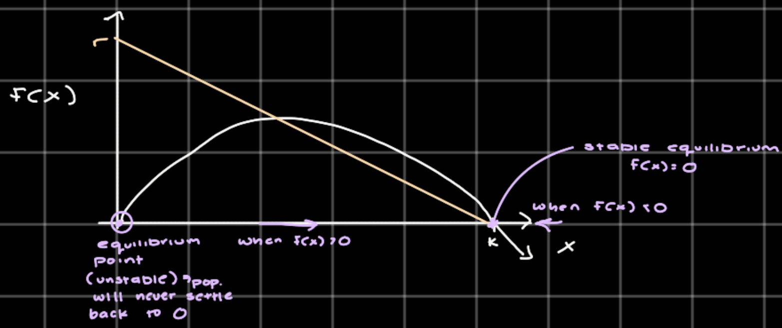
$\dot{x} = F(x) = r\left(1 - \frac{x}{K}\right)x \qquad r > 0,\ K > 0$

growth rate

carrying capacity

intrinsic growth rate





· For scalar fcns, stability can be determined by sign of $F(x)$ around the equilibrium

· equilibrium : $x = x^*$ s.t. $F(x^*) = 0$

$\rightarrow x^* = 0 \qquad x^* = K$

$x \begin{cases} \in (0, K) & f(x) > 0 \\ > K & f(x) < 0 \end{cases}$

$\Rightarrow x = 0$ is unstable equilibrium

$x = K$ is asymptotically

Linearization

· local stability properties of x^* can be determined by linearizing the vector field $F(x)$ at x^*

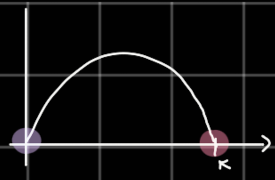
· 1st order Taylor series approx:

$F(x^* + \tilde{x}) = \underbrace{F(x^*)}_{=0} + \underbrace{\frac{\partial F}{\partial x}\Big|_{x=x^*}}_{A} \tilde{x} + \text{HOT}$

$\dot{\tilde{x}} = A\tilde{x}$ } linear system

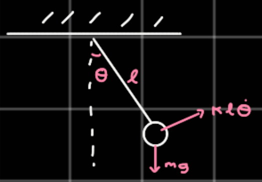
↳ zeroing our states around the equilibrium

$\tilde{x} = x - x^*$



- If $\forall \lambda_i$ $Re\{\lambda_i(A)\} < 0$, x^* is asymptotically stable
- If $\exists \lambda_i$ $Re\{\lambda_i(A)\} > 0$, x^* is unstable

Example



- pendulum w/ damping coefficient k
- dynamics:

$$\ell m \ddot{\theta} = -k\ell\dot{\theta} - mg\sin\theta$$

$$\ddot{\theta} = -\frac{k}{m} - \frac{g}{\ell}\sin\theta$$

Euler-Lagrange Equations of motion:

- Lagrangian: $\mathcal{L}(\theta, \dot{\theta}) = KE - PE$

$$= \frac{1}{2}m(\ell\dot{\theta})^2 - mg\ell\cos\theta$$

- d'Alembert Principle: $\frac{d}{dt}\left(\frac{d\mathcal{L}}{d\dot{\theta}}\right) - \frac{d\mathcal{L}}{d\theta} = \tau_{ext}$

$$\frac{d}{dt}(m\ell^2\dot{\theta}) + mg\ell\sin\theta = -k\ell^2\dot{\theta}$$

$$m\ell^2\ddot{\theta} + mg\ell\sin\theta = -k\ell^2\theta$$

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = -\frac{k}{m}\theta$$

$$\ddot{\theta} = -\frac{k}{m}\theta - \frac{g}{\ell}\sin\theta$$

Define state $x = \begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}x_1 \\ x_2\end{bmatrix}$, $\dot{x} = \begin{bmatrix}\dot{\theta} \\ \ddot{\theta}\end{bmatrix}$

$$\dot{x} = \begin{bmatrix} x_2 \\ -\frac{k}{m}x_2 - \frac{g}{\ell}\sin x_1 \end{bmatrix}$$

x^* when $\dot{x} = 0$

$\Rightarrow$ 2 equilibrium points: $\underbrace{(0,0)}_{x_1^*}$ & $\underbrace{(\pi,0)}_{x_2^*}$

$$\frac{df}{dx} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell}\cos x_1 & -\frac{k}{m} \end{bmatrix}$$

$$\left.\frac{df}{dx}\right|_{x_1^*} = \begin{bmatrix} 0 & 1 \\ -g/\ell & -k/m \end{bmatrix}$$ ← stable equilibrium bc all $\lambda_i < 0$

$$\left.\frac{df}{dx}\right|_{x_2^*} = \begin{bmatrix} 0 & 1 \\ g/\ell & -k/m \end{bmatrix}$$ ← unstable bc $\lambda_i > 0$

Phase Portrait