# Lecture 10: Lyapunov's Linearization

Lyapunov Equation: $A^T P + PA = -Q$

* There's no computational advantage to checking $\lambda_i$ of A

$f(x) \approx Ax + g(x)$

- We will establish $V = x^T P x$ that is locally a Lyapunov fcn for the nonlinear system

## Lyapunov's Linearization Method:

$\dot{x} = f(x)$ $\quad$ $f(0) = 0$

$f: D \to \mathbb{R}^n$, $C^1$ $\quad$ $D \subset \mathbb{R}^n$ is neighborhood of $x = 0$

can transform $\to$ $\dot{x} = Ax + g(x)$

Mean Value Theorem (MVT): $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

$$\frac{\partial f_i}{\partial x}(z_i) = \frac{f_i(x) - f_i(0)}{x - 0}$$

$z_i \in (0, x)$

$$f_i(x) = \cancel{f_i(0)} + \frac{\partial f_i}{\partial x}(z_i)\, x$$

$$= \underbrace{\frac{\partial f_i}{\partial x}(0)}_{A}\, x + \underbrace{\left[\frac{\partial f}{\partial x}(z_i) - \frac{\partial f_i}{\partial x}(0)\right]}_{g(x)}$$

$$= Ax + g(x)$$

$g_i(x)$ satisfies $\quad |g_i(x)| \leq \left|\frac{\partial f_i}{\partial x}(z_i) - \frac{\partial f_i}{\partial x}(0)\right| |x|$

$\Rightarrow$ in a small neighborhood of $x = 0$

$\Rightarrow g_i(x) \to 0$

$$\frac{|g(x)|}{|x|} \to 0 \quad \text{as} \quad \|x\| \to 0$$

↳ Lipschitz continuity also gives us this bound

Theorem: Let $x=0$ be an equilibrium point for the nonlinear system

$$\dot{x} = F(x)$$

$F : D \to \mathbb{R}^n$ $C^1$ $D \subset \mathbb{R}^n$ is n of $x=0$

$$\text{Let } A = \left.\frac{\partial F(x)}{\partial x}\right|_{x=0}$$

Then:

1. If $Re\{\lambda_i(A)\} < 0$ $\forall \lambda_i$ then

   $\Rightarrow$ $x=0$ is asymptotically stable for the nonlinear system

2. If $\exists \lambda_i$ s.t. $Re\{\lambda_i(A)\} > 0$

   $\Rightarrow$ $x=0$ is unstable for the nonlinear system

Note: We can conclude only asymptotic stability

$\to$ inconclusive if A has eigenvalue on the imaginary axis

Proof:

Find $P = P^T \succ 0$ ("Lyapunov Matrix") st

$$A^T P + PA = -Q < 0$$

use $V(x) = x^T P x$ as a Lyapunov fcn for

$$\dot{x} = F(x) = Ax + g(x)$$

$$\dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x$$

$$= x^T P(Ax + g(x)) + (Ax + g(x))^T P x$$

$$= x^T \underbrace{(PA + A^T P)}_{-Q} x + 2x^T P g(x)$$

$$= -x^T Q x + 2x^T P g(x)$$

Since $\frac{|g(x)|}{|x|} \to 0$ as $x \to 0$

We can find a ball st $2x^T P g(x) < 0$

For any $\gamma > 0$, $\exists r > 0$ s.t

$$|g(x)| < \gamma |x| \quad \forall |x| < r$$


$$\dot{V}(x) = -x^T Q x + 2 x^T P g(x)$$ $\|g(x)\| < \gamma \|x\|$

$$\leq -x^T Q x + 2\gamma |P| |x|^2$$

Aside:

$$\lambda_{min}(Q) |x|^2 \leq x^T Q x \leq \lambda_{max}(Q) |x|^2$$

$$\dot{V}(x) < [-\lambda_{min}(Q) + 2\gamma |P|] |x|^2$$

We choose

$$\gamma < \frac{\lambda_{min}(Q)}{2 \|P\|}$$

$\Rightarrow \dot{V}(x) < 0$ in a ball of radius $r(\gamma)$

* The rest follows from Lyapunov stability Thm

## Region of Attraction

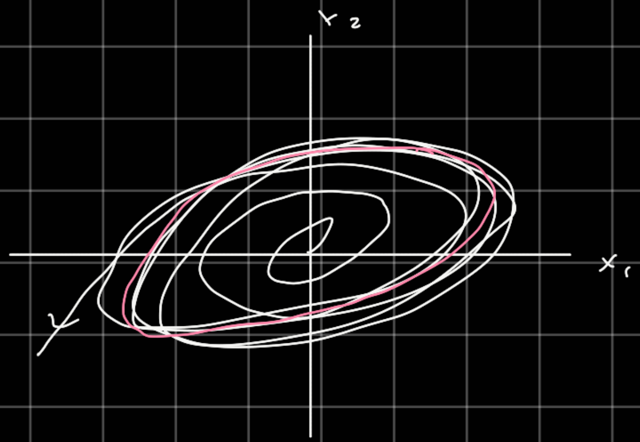
$$R_A = \{x : \phi(t, x) \to 0\}$$

$\uparrow$ flow of system $x(t)$

- $R_A$ "quantifies" local asymptotic stability
- Global asymptotic stability: $R_A = \mathbb{R}^n$

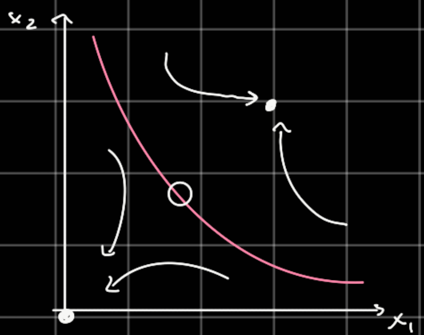
Proposition: If $x = 0$ is asymptotically stable, then its region of attraction is an open, connected, invariant set & boundary is formed by trajectories

Van der Pol System (in reverse time $\tau = -t$)




Note: a limit cycle is an isolated periodic orbit

Bistable switch

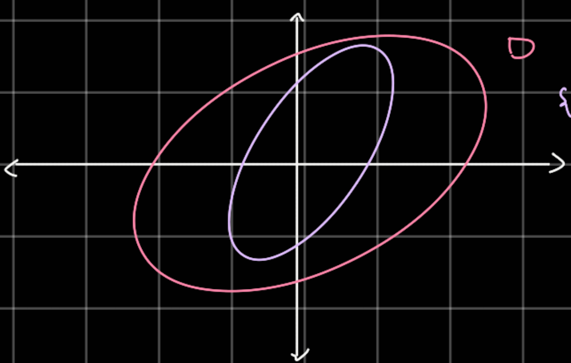



if $ab < 0.5$

Establishing $R_A$ with $V(x)$

→ Suppose $\dot{V}(x) < 0$ in $D - \{0\}$

⇒ the level sets of $V$ inside $D$ are invariant & trajectories starting in $\Omega_c$ converge to the origin



$D$ where $\dot{V}(x) < 0$

$\{x \mid V(x) \leq c\} \subset R_A$

* this approximation depends on choice of $V(x)$
* an often conservative (under approximation) is $V(x) = x^T PA$

$PA + A^T P = -Q$

example: Van der Pol system

$t = -t$

$\dot{x}_1 = -x_2$

$\dot{x}_2 = x_1 + (x_1^2 - 1)x_2$

Linearize: $A = \frac{\partial f}{\partial x}\Big|_{x=0} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$

$\lambda_{1,2} = -0.5 \pm 0.866i$

$\Rightarrow$ we can find $P$ s.t

$A^T P + PA = -Q$

$\rightarrow$ Pick $Q = I$

$\Rightarrow$ solution to $P = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1 \end{bmatrix}$

$\Rightarrow$ $V(x) = x^T P x$ is a Lyapunov Fcn for $\dot{x} = f(x)$ in some neighborhood

We want to find the largest set $\Omega_c = \{V(x) \le c\}$ s.t. $\dot{V}(x) < 0$

$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x}$

$= [-x_2 \quad x_1 + (x_1^2 - 1)x_2] \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$= -(x_1^2 + x_2^2) - (x_1^3 x_2 - 2x_1^2 x_2^2)$

$\le -|x|_2^2 + |x_1||x_1 x_2|\underbrace{|x_1 - 2x_2|}_{\le \sqrt{5}|x_2|}$

$\rightarrow |x_1| \le |x_2|$

$|x_1 x_2| \le \frac{|x_2|^2}{2}$

$\Rightarrow \dot{V}(x) \le -|x|_2^2 + \frac{\sqrt{5}}{2}\|x\|_2^2$

$\dot{V}(x) < 0$ w/in a ball of $r^2 = \frac{2}{\sqrt{5}} = 0.8944$

$\Rightarrow$ we can find a level set $\Omega_c \subset B_r(0)$

$\min V(x) = \lambda_{min}(P) r^2$

$V(x) = x^T P x$

$$c = 0.69\,(0.8944)$$

$$= 0.6171$$