Lecture 2: Nonlinear Phenomena

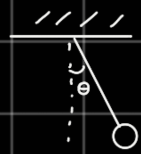
Review

Pendulum example:

"assume force resisting motion of the pendulum, proportional to the speed of the mass"

$$F_{ext} = -kv$$
$$= -kl\dot{\theta}$$

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = \tau_{ext}$$



$$ml^2\ddot{\theta} + mgl\sin\theta = -kl^2\dot{\theta}$$

$$\ddot{\theta} = -\frac{k}{m}\dot{\theta} - \frac{g}{l}\sin\theta$$

$$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \dot{x} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{k}{m}x_1 - \frac{g}{l}\sin x_1 \end{bmatrix}$$

Steps:

1. What are the equilibrium points?

$$\dot{x}^* = f(x^*) = 0$$

$$x^* = \{ \underbrace{(0, 0)}_{x_1^*}, \underbrace{(\pi, 0)}_{x_2^*} \}$$

2. Linearize our system about x^*.

$$\dot{x} = Ax$$

$$\left.\frac{\partial F}{\partial x}\right|_{x^*}$$

Recall Jacobian

$$\frac{\partial F}{\partial x} := \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{bmatrix}$$

$$\frac{\partial F}{\partial x} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos x_1 & -\frac{k}{m} \end{bmatrix}$$

$$\left.\frac{\partial F}{\partial x}\right|_{x^* = x_1^*} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & \frac{k}{m} \end{bmatrix} \rightsquigarrow \mathrm{Re}(\lambda_i(A)) < 0$$

$\Rightarrow$ stable!

Recall
- → negative eigenvalues imply stability
- → positive eigenvalues imply instability

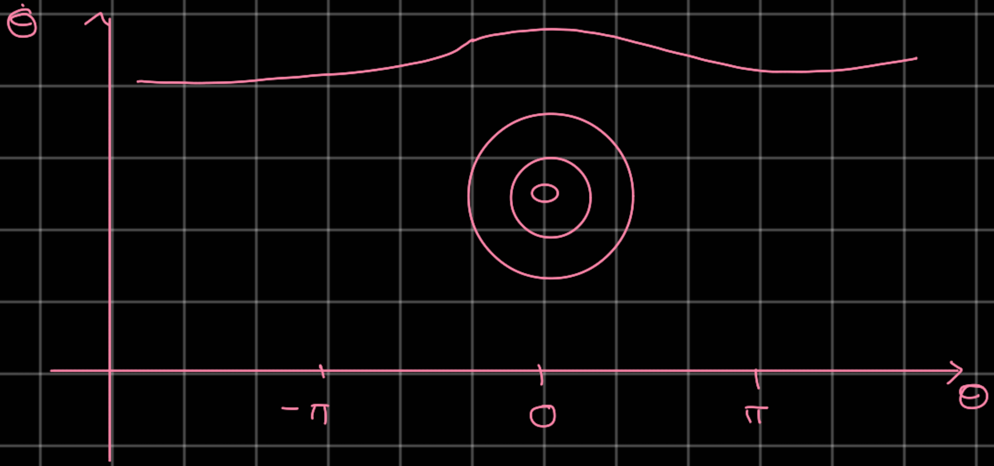
$$\left.\frac{\partial F}{\partial x}\right|_{x^* = x_1^*} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{k}{m} \end{bmatrix} \rightarrow \begin{matrix} \mathrm{Re}(\lambda_1(A)) < 0 \\ \mathrm{Re}(\lambda_2(A)) > 0 \end{matrix} \Rightarrow \text{unstable}$$

Takeaway: Linearization is a valid approach for analyzing the behavior of the nonlinear systems

But the limitations are that

I. only valid locally

II. nonlinear dynamics are richer



Essential Nonlinear Phenomena

① Finite Escape Time

(the state x goes to ∞ in ∞ time)

Example $\dot{x} = x^2$ ↳ For a linear system, this would never happen

$$\dot{x} = \frac{dx}{dt} = x^2$$

$$\frac{1}{x^2}\,dx = dt$$

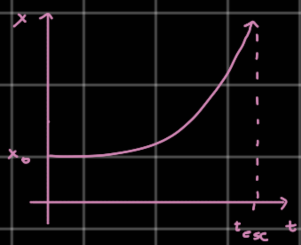
$$\int \frac{1}{x^2}\,dx = \int dt$$

$$-\frac{1}{x} = t + c$$

$$x(t) = \frac{1}{c - t}, \quad x(0) = \frac{1}{c}$$

$$x(t) = \frac{1}{\frac{1}{x_0} - t}$$

$$t_{escape} = \frac{1}{x_0}$$



② Multiple Isolated Equilibria

Linear system:

- either unique x^* or continuum

Pendulum: two isolated x^*

"multi-stable" systems

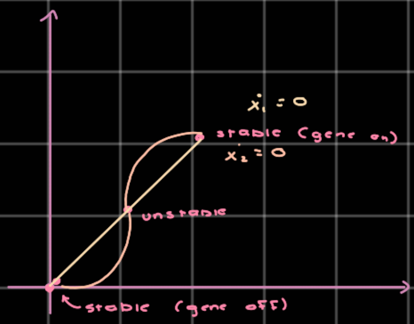
Example: Bi-stable switching

$$\dot{x}_1 = -a x_1 + x_2 \quad \leftarrow \text{concentration of protein}$$

$$\dot{x}_2 = \frac{x_1^2}{1 + x_1^2} - b x_2 \quad \leftarrow \text{mRNA}$$

$a > 0$, $b > 0$

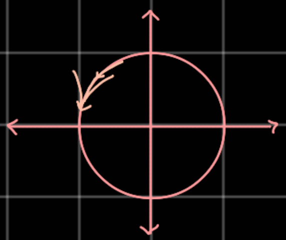
→ gene stimulation by transcription $\frac{x_1^2}{1+x_1^2}$



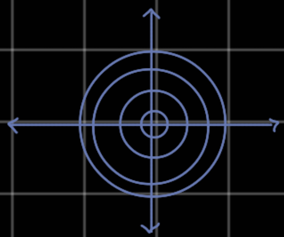
③ Limit cycles ⇝ "like a trajectory of equilibrium points, in state space"

- linear oscillators ⇝ $\underbrace{\text{continuum}}_{\text{every "circle"}}$ of periodic orbits

$\dot{x} = Ax \quad A = \begin{bmatrix} 0 & -\beta \\ \beta & 0 \end{bmatrix}$

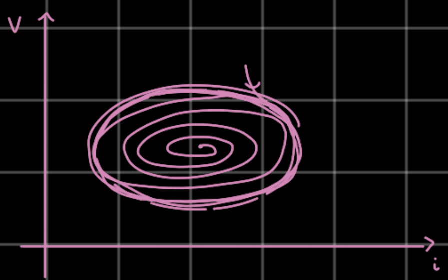


Nonlinear
↳ idea



linear analogy (harmonic oscillator)

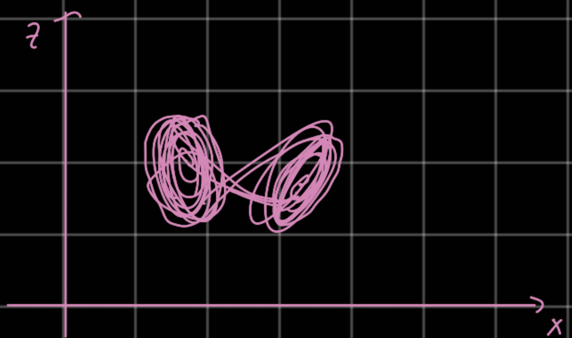
Example : van der Pol oscillator
→ system w/ self-sustained oscillations



④ Chaos : irregular oscillations

(never exactly repeating)

Example : Lorenz System



→ For a continuous-time, time-invariant system
⇝ need $n \geq 3$ chaos

$\underline{n=2}$ Poincaré - Bendixson Theorem : guarantees regular behavior
↳ only applies to time-invariant systems

→ for time-varying, only need $n \geq 2$

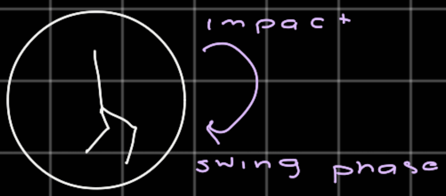
→ For discrete time

$$\delta x_k, \quad n \geq 1 \quad \text{is sufficient for chaos}$$

(bc of lack of continuity assumptions)

(5) Multiple Modes of Behavior (Hybrid systems)

Ex: walking!



$$\dot{x} = f(x) + g(x)u$$

$$\mathcal{H} = \begin{cases} \dot{x} = f(x) + g(x)u & x \notin S \\ x^+ = \Delta(x) & x \in S \end{cases}$$

Planar (second-order) Dynamical Systems

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

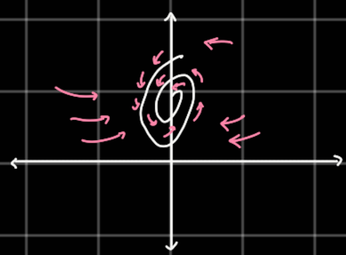
$$\dot{x}_1 = f_1(x_1, x_2)$$

$$\dot{x}_2 = f_2(x_1, x_2)$$

- solution trajectories are "curves" in the plane
  - either a phase plane or a state plane

$$x(t) = (x_1(t), x_2(t))$$

- family of all trajectories (solution curves) is called the phase portrait
  - ↳ $f(x)$ is the vector field on the plane



→ Let's look at phase portraits for linear systems!

$$\dot{x} = Ax$$

Jordan Form : $J = T^{-1}AT$

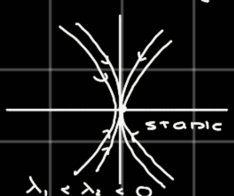
change of coordinates: $z = T^{-1}x$

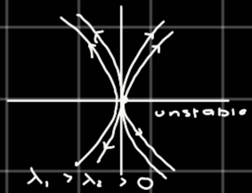
3 options for the Jordan form :

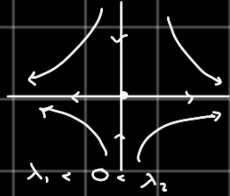
(1) $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ D

$$\dot{z}_1 = \lambda_1 z_1$$

$$\dot{z}_2 = \lambda_2 z_2$$







$$, \begin{bmatrix} \lambda & \kappa \\ 0 & \lambda \end{bmatrix} , \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$