Lecture 3: Phase Portraits Near Hyperbolic Equilibria

Overview:

- Hartman-Grobman Thm
- Bendixson's Thm
- Invariant sets

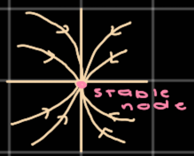
Review: Phase portraits of linear system

$\dot{x} = Ax$

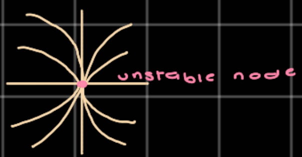
$J := T^{-1}AT$

① Distinct, real

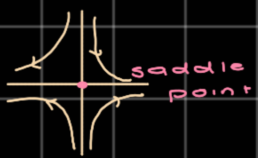
$\lambda_1 < \lambda_2 < 0 \rightarrow J = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ · equilibrium point called a node



$\lambda_1 > \lambda_2 > 0 \rightarrow J = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ · equilibrium point called an unstable node



$\lambda_2 > 0 > \lambda_1 \rightarrow J = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ · equilibrium point called a saddle point



② Complex eigenvalues

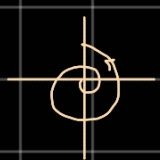
$$\lambda_{1,2} = \alpha \pm \beta j \rightarrow J = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

$$\dot{z}_1 = \alpha z_1 - \beta z_2 \xrightarrow{\text{polar coords}} \dot{r} = ar$$

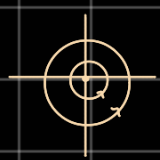
$$\dot{z}_2 = \alpha z_2 + \beta z_1 \longrightarrow \dot{\theta} = \beta$$

$\alpha < 0$

$\alpha > 0$



$\alpha = 0$



[Note: these phase portraits assume $\beta > 0$ s.t. direction of rotation is ccw]

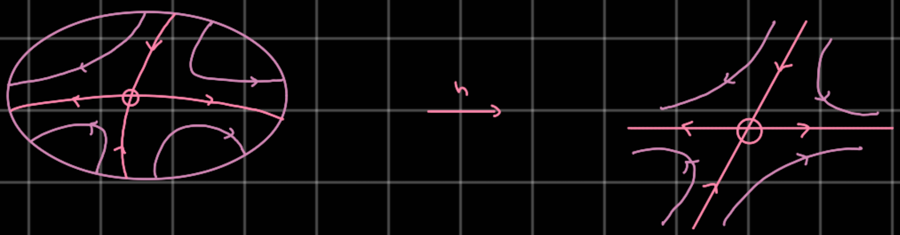
Phase Portraits Near Hyperbolic Equilibria

- hyperbolic equilibrium: linearization has no eigenvalues on the imaginary axis
  - ↳ no oscillatory behavior

Thm: Hartman - Grobman

- "a continuous deformation" maps one phase portrait to another
  "nonlinear" → "linear"

intuition



↳ eg Mercator projection!

- If x^* is a hyperbolic equilibrium of $\dot{x} = f(x)$, $x \in \mathbb{R}^n$, then $\exists$ a homeomorphism $z = h(x)$ defined in a neighborhood of x^* that maps trajectories of $\dot{x} = f(x)$ to those of $\dot{z} = Az$

  where $A \triangleq \left.\frac{\partial f}{\partial x}\right|_{x=x^*}$

  example:

  $$\dot{x}_1 = -x_2 + a x_1 (x_1^2 + x_2^2) \qquad \Rightarrow \qquad \dot{r} = a r^3$$

  $$\dot{x}_2 = x_1 + a x_2 (x_1^2 + x_2^2) \qquad \qquad \dot{\theta} = 1$$

  $$x^* = (0, 0) \qquad A = \left.\frac{\partial f}{\partial x}\right|_{x=x^*} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

  ⇒ There is no continuous deformation that maps the phase portrait of the linearization to that of the original nonlinear model

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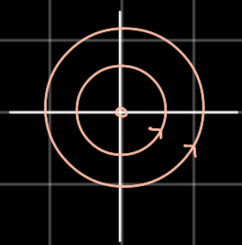


$\dot{x} = Ax$

linear

$\dot{x} = f(x)$

$a > 0$

nonlinear

Periodic Orbits in the Plane

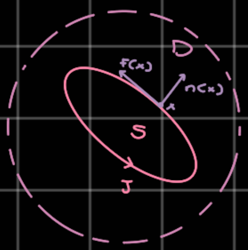
Thm: Bendixson's Theorem

- For a time-invariant planar system

$$\dot{x}_1 = F_1(x_1, x_2) \qquad \dot{x}_2 = F_2(x_1, x_2)$$

if $\nabla \cdot F(x) = \dfrac{\partial F_1}{\partial x_1} + \dfrac{\partial F_2}{\partial x_2}$ is not identically 0 & does not change sign in a simply connected region D, then there are no periodic orbits lying entirely in D.

Pf (By contradiction):



[Note that this only applies on the periodic orbit]

Suppose a periodic orbit J lies in D.
Let S denote the region enclosed by J & $n(x)$ the normal vector to J at x.

Then, $F(x) \cdot n(x) = 0 \quad \forall x \in J$. By Divergence Thm:

$$\underbrace{\int_J F(x) \cdot n(x)\, dl}_{=0} = \underbrace{\iint_S \overbrace{\nabla \cdot f(x)}^{\text{Divergence}}\, dx}_{\neq 0}$$

Path integral — Volume integral

bc
- $\nabla \cdot F(x) \not\equiv 0$ and
- $F(x)$ doesn't change sign

(these are the 2 conditions for having a periodic orbit)

→ intuition, if you change the volume of somethin (eg adding water to a balloon), its path will also change

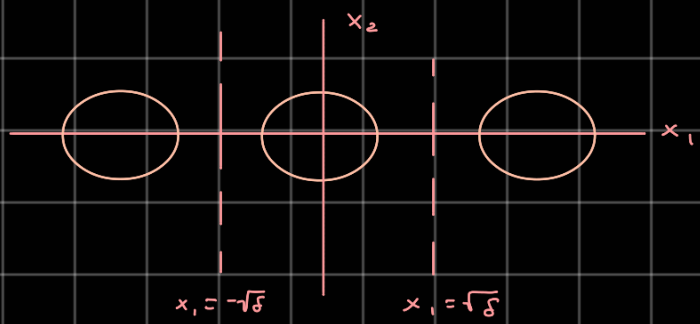
example:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\delta x_2 + x_1 - x_1^3 + x_1^2 x_2 \qquad \delta > 0$$

$$\nabla \cdot F(x) = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} = x_1^2 - \delta$$

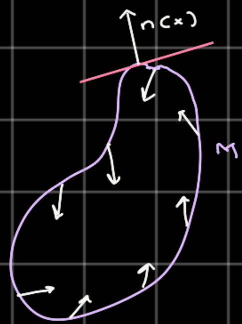
based on value of δ, we can say where our periodic orbits can/can't lie



Therefore, no . . . .

Invariant Sets

- Notation: $\varphi(t, x_0)$ denotes trajectory of $\dot{x} = F(x)$ with initial condition $x(0) = x_0$
- Def: a set $M \subset \mathbb{R}^n$ is positively (forward) invariant if for each $x_0 \in M$, $\varphi(t, x_0)$ $\forall t \geq 0$
- Def: a set $M \subset \mathbb{R}^n$ is negatively (backward) invariant if for each $x_0 \in M$, $\varphi(t, x_0)$ $\forall t \leq 0$



system dynamics point inwards

If $F(x) \cdot n(x) \leq 0$ on the boundary then M is positively invariant

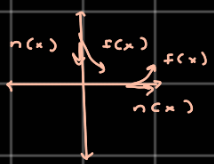
example: Lotka-Volterra equations (predator-prey)

$\dot{x} = (a - by)x$ prey (exp. growth when $y = 0$)

$\dot{y} = (cx - d)y$ predator (exp. decay when $x = 0$)

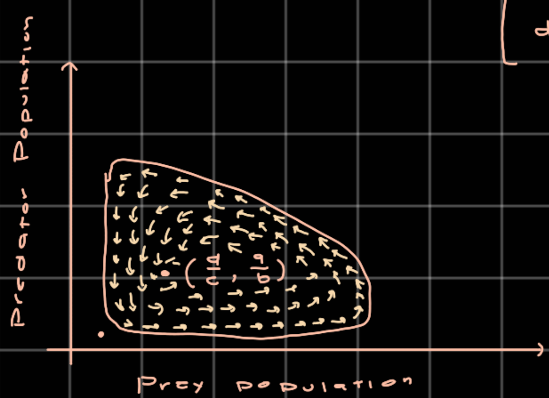
$a, b, c, d > 0$

non-neg. quadrant is invariant:



x-axis: $\begin{bmatrix} ax \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \end{bmatrix} = 0$

y-axis: $\begin{bmatrix} 0 \\ dy \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \end{bmatrix} = 0$



hue

hi!