Lecture 4: Periodic Orbits in the Plane

Overview:

- Bendixsons Thm
- Poincaré - Bendixson Thm
- Index Theory

Remarks:

① The divergence of a continuously differentiable vector field $f(x)$ for $x \in \mathbb{R}^n$ is defined as:

$$\operatorname{div}(f(x)) = \nabla \cdot f(x) = \left(\frac{\partial}{\partial x_1}, \dots \frac{\partial}{\partial x_n}\right) \cdot (f_1, \dots, f_n) = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}$$

② Hyperbolic Equilibria $\Leftrightarrow$ eigs of A w/ no real part

Periodic Orbits in the Plane

- Two conditions:

  ① Bendixson's

  (absence of periodic orbits)

  $\rightarrow$ if $\operatorname{div}(f(x)) \neq 0$ $\forall x \in D$ & does not change sign,

  then D contains no periodic orbits

  ② Poincaré - Bendixson ⟵ only apply to things in the plane

  (existence of periodic orbits)

  Suppose M is compact (closed & bounded) and positively invariant for the planar, time invariant system $\dot{x} = f(x)$ $x \in \mathbb{R}^n$.

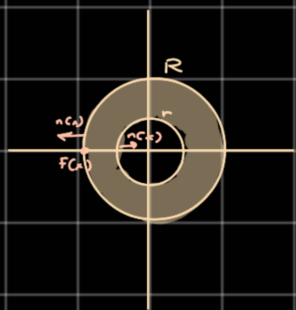
  If M contains no equilibrium points

  $\Rightarrow$ it contains a periodic orbit

example of PCB : Harmonic oscillator

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\dot{x}_1 = -x_2$$
$$\dot{x}_2 = x_1$$



dynamics

For $R > r > 0$, the ring

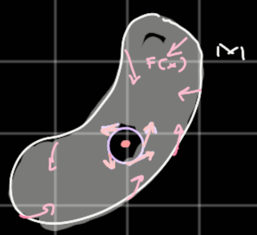
$$\{x : r^2 \leq x_1^2 + x_2^2 \leq R^2\}$$

is compact, invariant, contains no equilibri[a]
$\Rightarrow$ at least 1 periodic orbit
(we know there are ∞ly many in this case)

If "no equilibrium" condition in the PB thm can be relaxed as

"If M contains one equilibrium which is an unstable focus or unstable node"

pf sketch:



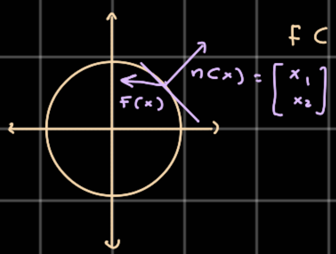
"carving out the interior"

Since the equilibrium is an unstable focus/node, we can encircle it with a small closed curve on which $f(x)$ points outward. Then, the set obtained from M by carving out the interior of the closed curve is positively invariant & contains no equilibrium

example

$$\dot{x}_1 = x_1 + x_2 - x_1(x_1^2 + x_2^2)$$
$$\dot{x}_2 = -2x_1 + x_2 - x_2(x_1^2 + x_2^2)$$

→ show $B_r \triangleq \{x \mid x_1^2 + x_2^2 \leq r^2\}$ is positively invariant for sufficiently large r (ie $f(x) \cdot n(x) \leq 0$)



$$f(x) \cdot n(x) = \begin{bmatrix} \dot{x}_1 = x_1 + x_2 - x_1(x_1^2 + x_2^2) \\ \dot{x}_2 = -2x_1 + x_2 - x_2(x_1^2 + x_2^2) \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= x_1^2 + x_1x_2 - x_1^2(x_1^2 + x_2^2) - 2x_1x_2 + x_2^2 - x_2^2(x_1^2 + x_2^2)$$

$$= -\underbrace{x_1x_2} + (x_1^2 + x_2^2) - (x_1^2 + x_2^2)^2$$

recall $|2x_1x_2| \leq x_1^2 + x_2^2$

$$\leq \frac{1}{2}(x_1^2 + x_2^2) + (x_1^2 + x_2^2) - (x_1^2 + x_2^2)^2$$
$$= \frac{3}{2}r^2 - r^4$$

$\Rightarrow$ $f(x) \cdot n(x) \leq \frac{3}{2}r^2 - r^4 \leq 0$ if $r^2 \geq \frac{3}{2}$

$\Rightarrow$ B_r positively invariant for $r \geq \sqrt{\frac{3}{2}}$ and contains the equilibrium $x = 0$

$$\left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \qquad \lambda_{1,2} = 1 \pm j\sqrt{2} \quad \text{unstable focus}$$

$\Rightarrow$ B_r must contain a periodic orbit

A more general form of PB Thm states that for time-inv. planar systems, bounded trajectories converge to equilibria, periodic orbits, or unions of equilibria connected by trajectories.

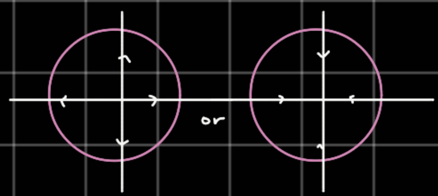
Corollary: No chaos for time-invariant planar systems

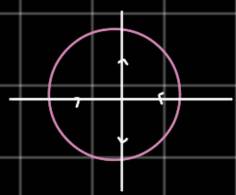
has parallels to winding theory

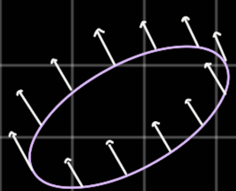
Index Theory (only applicable to planar systems)

- index: The index of a closed curve is K if, when traversing the curve in one direction, $F(x)$ rotates by $2\pi K$ in the same direction. The index of an equilibrium is defined to be the index of a small curve around it that doesn't enclose another equilibrium.

type of equilibrium /curve	index
• node, focus, center	+1 one rotation in same direction
• saddle	-1 as you follow the curve,
• any closed orbit	+1
• a closed curve not enclosing any equilibria	0







Thm: index of a closed curve is equal to the sum of the indices of the equilibria inside

Corollary: Inside any periodic orbit there must be at least one equilibrium and the indices of the equilibria enclosed must add up to +1