Lecture 5: Bifurcations

- a bifurcation is an abrupt change in qualitative behavior as a behavior is varied
  ↳ determined by the pattern of its equilibrium points & periodic behavior, as well as by their stability properties
  - need to pay attention to stability in the presence of infinitesimally small perturbations

- ex of bifurcations: equilibria / limit cycles appearing/dissapearring, becoming stable/unstable.

Fold Bifurcation

aka "saddle node" / "blue sky" bifurcation

example

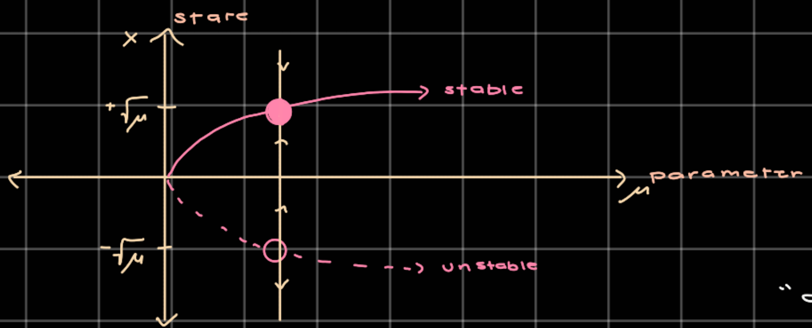
$\dot{x} = \mu - x^2$

if $\mu > 0$, 2 equilibria: $x = \pm\sqrt{\mu}$

if $\mu < 0$, no equilibria

Note: bifurcation diagrams sketch the amplitude of the equilibrium points as a function of the bifurcation param. Solid lines represent stable nodes/foci/limit cycles. Dashed lines represent unstable nodes/foci/limit cycles

Bifurcation diagram



"dangerous"

Transcritical Bifurcation

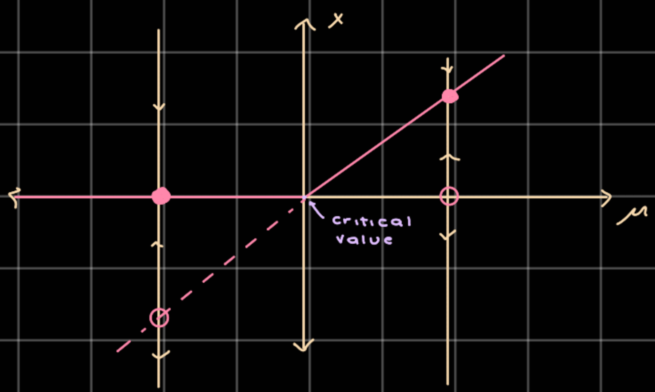
example $\dot{x} = \mu x - x^2$

$= x(x - \mu)$

equilibria: $x^* = 0$, $x^* = \mu$

$$\frac{\partial f}{\partial x} = \mu - 2x = \begin{cases} \mu & \text{if } x = 0 \\ -\mu & \text{if } x = \mu \end{cases}$$

	if $\mu > 0$	$\mu < 0$
$x = 0$ ←	unstable	stable
$x = \mu$	stable	unstable



"soft"

Pitchfork Bifurcation

ex 1

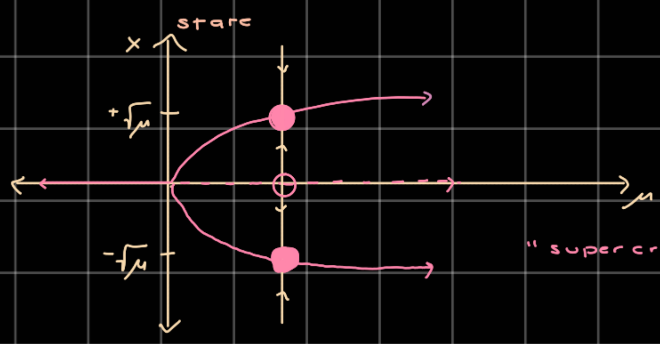
$\dot{x} = \mu x - x^3$

$= x(\mu - x^3)$

$x^* = 0 \qquad x^* = \pm\sqrt{\mu}$ for $\mu > 0$

$$\frac{\partial f}{\partial x} = \mu - 3x^2 = \begin{cases} \mu \\ -2\mu \end{cases}$$

	if $\mu < 0$	if $\mu > 0$
μ	stable	unstable
-2μ	N/A	stable



"supercritical pitchfork"

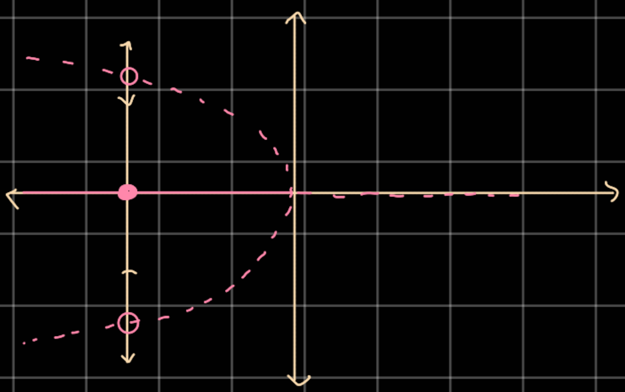
ex2 : subcritical pitchfork

$\dot{x} = \mu x + x^3$

$x^* = 0 \quad \forall \mu$

$x^* = \pm\sqrt{-\mu}$ if $\mu < 0$

	$\mu < 0$	$\mu > 0$
$\left.\dfrac{\partial f}{\partial x}\right\vert_{x=0} = \mu$	stable	unstable
$\left.\dfrac{\partial f}{\partial x}\right\vert_{x=\pm\sqrt{-\mu}} = -2\mu$	unstable	N/A

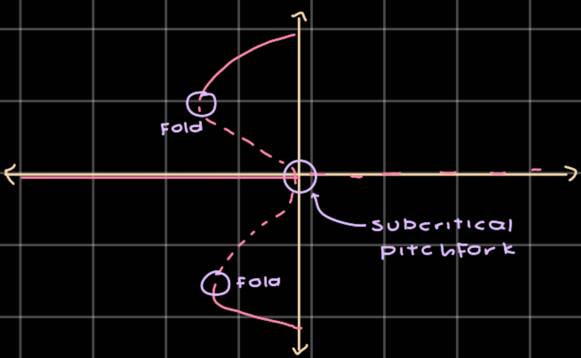


dangerous

"subcritical pitchfork"

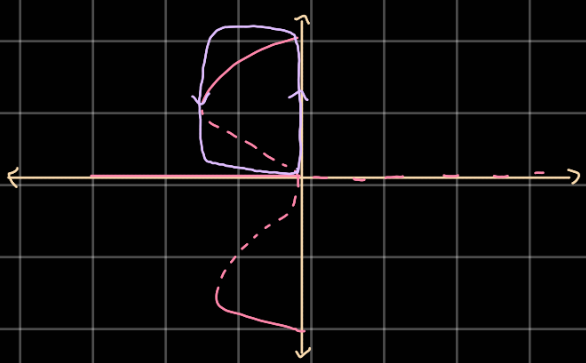
ex 3

$\dot{x} = \mu x + x^3 - x^5$



ex 4:

hysteresis arising from a subcritical pitchfork bifurcation



Higher-Order Systems

- fold, transcritical & pitchfork bifurcations, as evident from the first-order examples above. They occur in higher-order systems too but are restricted to a one-dim manifold

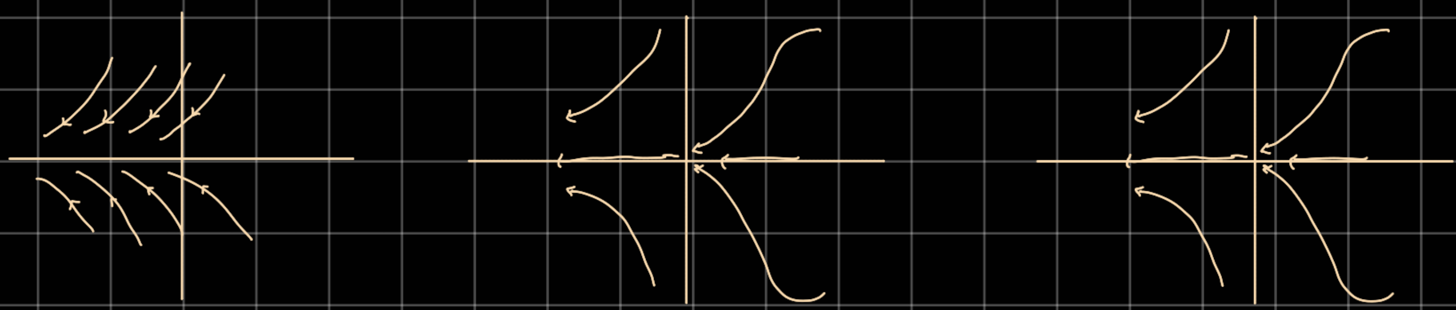
1D subspace: $c_1^T x = \cdots = c_{n-1}^T x = 0$

1D manifold: $g_1(x) = \cdots = g_{n-1}(x) = 0$

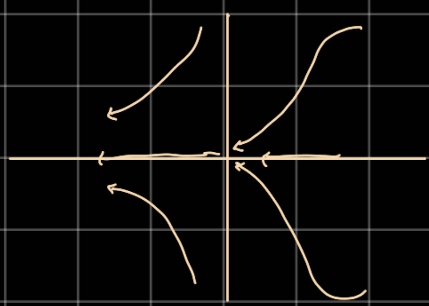
ex 1:

$$\dot{x}_1 = \mu - x_1^2$$

$$\dot{x}_2 = -x_2$$



Khalil Fig 2.27

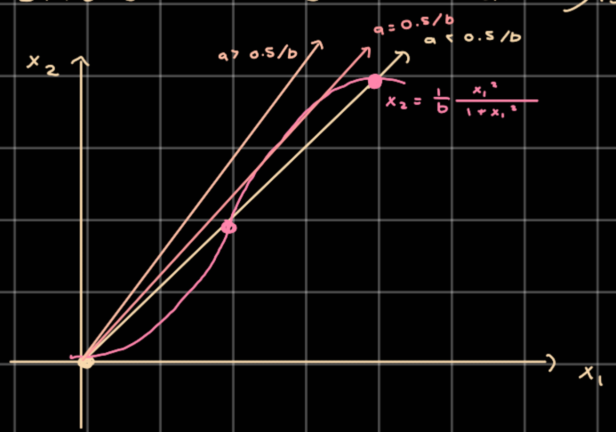


Example 2 : bistable switch

$$\dot{x}_1 = -a x_1 + x_2$$

$$\dot{x}_2 = \frac{x_1^2}{1 + x_1^2} - b x_2$$

Fold bifurcation occurs at $\mu \triangleq ab = 0.5$



Characteristic of 1D bifurcations:

$$\left.\frac{\partial f}{\partial x}\right|_{\mu=\mu_c,\ x=x^*(\mu_c)}$$ has an eigenvalue at 0

Hopf Bifurcation