Lecture 6: Center Manifold Theory

Overview:

- Center Manifold Theory
- Discrete-Time System
- Chaos in discrete time

Motivation

- center manifold theory is used to study systems where linearization fails

Theorem 4.7 (Khalil)

Let $x = 0$ be an equilibrium point for

$$\dot{x} = f(x)$$

where $f : D \to \mathbb{R}^n$ continuously differentiable and D is a neighborhood of the origin.

Let $A = \left. \frac{\partial f}{\partial x}(x) \right|_{x=0}$

Then,

1. $x^* = 0$ is asymptotically stable if $Re\{\lambda_i\} < 0$ $\forall \lambda_i(A)$
2. $x^* = 0$ is unstable if $Re\{\lambda_i\} > 0$ for some $\lambda_i(A)$

Note: If A has some λ_i's with zero real parts ($Re\{\lambda_i\} = 0$) and the rest are negative ($Re\{\lambda_i\} < 0$) then linearization fails

Lets assume k eigenvalues s.t. $Re\{\lambda_i\} = 0$

m " " $Re\{\lambda_i\} < 0$

$\shortparallel$

$n-k$

Mathematical Preliminaries

- a k-dimensional manifold can be interpreted as

$$\eta(x) = 0$$

with $\eta : \mathbb{R}^n \to \underbrace{\mathbb{R}^m}_{\# \operatorname{Re}\{\lambda_i\} < 0}$ sufficiently smooth

example: unit circle

$$\{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}$$

is a one-dimensional manifold in $\mathbb{R}^2$

example: unit sphere

$$\{x \in \mathbb{R}^n \text{ s.t. } \sum_{i=1}^{n} x_i^2 = 1\}$$

is an $n-1$-dimensional manifold in $\mathbb{R}^n$

- a manifold is an invariant manifold if:

$$\eta(x(0)) = 0 \Rightarrow \eta(x(t)) \equiv 0 \quad \forall t \in [0, t_1) \subset \mathbb{R}$$

where $[0, t_1)$ is any time interval over which $x(t)$ is defined

Center Manifold Theory

$$\dot{x} = F(x) \qquad F(0) = 0$$

Suppose $A \triangleq \left.\frac{\partial F}{\partial x}\right|_{x=0}$ has k eigenvalues with zero real parts, and $m = n - k$ eigenvalues with negative real parts.

↳ Define $\begin{bmatrix} y \\ z \end{bmatrix} = Tx$ such that

$$TAT^{-1} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

where the eigenvalues of A_1 have zero real parts & the eigenvalues of A_2 have negative real parts.

↳ Rewrite $\dot{x} = F(x)$ in the new coordinates

$$\dot{y} = A_1 y + \underbrace{g_1(y,z)}$$
$$\dot{z} = A_2 z + \overbrace{g_2(y,z)}$$

error due to linearization

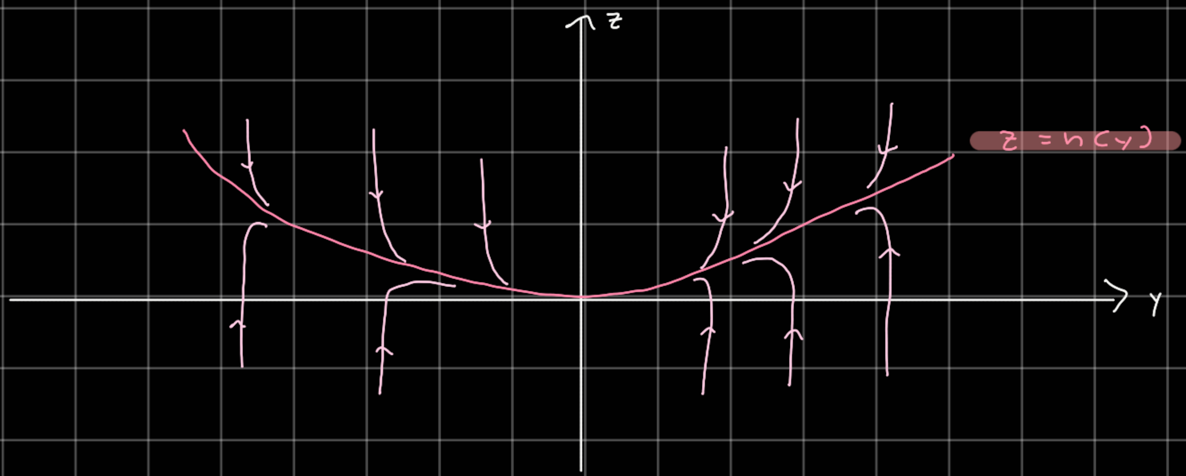
$g_i(0,0) = 0$ $\quad \frac{\partial g_i}{\partial y}(0,0) = 0$

$\frac{\partial g_i(0,0)}{\partial z} = 0$ $\quad i = 1, 2$

[Note: g_1 & g_2 inherit the properties of $\tilde{F}$ in eqn:
$\dot{x} = F(x) = Ax + \tilde{F}(x)$
with $\tilde{F}(x) = F(x) - \frac{\partial F(x)}{\partial x}\Big|_{x=0}$
which has the properties
$\tilde{F}(0) = 0$ & $\frac{d\tilde{F}}{dx}(0) = 0$]

Theorem 1: $\exists$ an invariant manifold $z = h(y)$ defined in a neighborhood of the origin s.t.

$$h(0) = 0 \qquad \frac{\partial h}{\partial y}(0) = 0$$



$z = h(y)$ is called a center manifold in this case

Reduced System

$$\dot{y} = A_1 y + g_1(y, h(y)) \qquad y \in \mathbb{R}^k$$

Theorem 2: If $y = 0$ is asymptotically stable for the reduced system, then $x = 0$ is asymptotically stable for the full system $\dot{x} = F(x)$.

On the other hand, if $y = 0$ is unstable for the reduced system, then $x = 0$ is unstable for the full system $\dot{x} = F(x)$.

Characterizing the Center Manifold

Define $w \triangleq z - h(y)$ & note that it satisfies

$$\dot{w} = \dot{z} - \frac{\partial h}{\partial y}\dot{y}$$

$$= A_2 z + g_2(y, z) - \frac{\partial h}{\partial y}(A_1 y + g_1(y, z))$$

The invariance of $z = h(y)$ means that $w = 0$ implies $\dot{w} = 0$ $\Rightarrow$ $\dot{w}$ must vanish when we substitute $z = h(y)$:

$$\boxed{A_2 h(y) + g_2(y, h(y)) - \frac{\partial h}{\partial y}(A_1 y + g_1(y, h(y))) = 0}$$

↳ To find $h(y)$, solve this PDE for h as a function of y

↳ if the exact solution is unavailable, an approximation might suffice

↳ For scalar y, expand $h(y)$ as:

$$h(y) = h_2 y^2 + \cdots + h_p y^p + O(y^{p+1})$$

where $h_1 = h_0 = 0$ $\quad \frac{\partial h}{\partial y}(0) = 0$

example 8.2 Khalil

$$\dot{y} = yz$$

$$\underbrace{\dot{z} = -z + ay^2}_{} \qquad a \neq 0$$

$g_1 = yz \qquad A_1 = 0$

$g_2 = ay^2 \qquad A_2 = -1$

$$h(x) = \underbrace{-h(y) + ay^2}_{\dot{z}} - \underbrace{\frac{\partial h}{\partial y}\, y h(y)}_{\frac{\partial h}{\partial y}\dot{y}} = 0$$

Try $h(y) = h_2 y^2 + O(y^3)$

$$0 = -h_2 y^2 - O(y^3) + ay^2 - (2h_2 y + O(y^2))\, y(h_2 y^2 + O(y^3))$$

$$= (a - h_2) y^2 + O(y^3)$$

$\Rightarrow h_2 = 9$

Reduced system: $\dot{y} = y(ay^2 + O(y^3))$

$= ay^3 + O(y^4)$

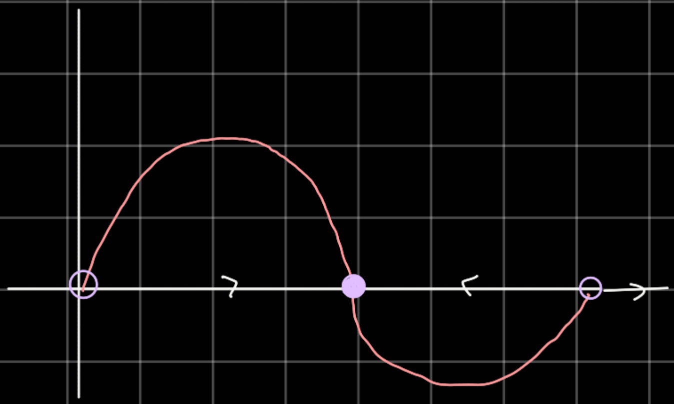
If $a < 0$, the full system is asymptotically stable

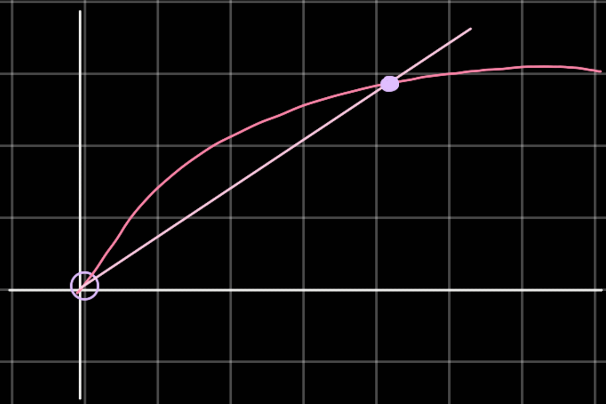
If $a > 0$, it's unstable

Discrete Time Models

CT: $\dot{x}(t) = f(x(t))$

$f(x^*) = 0$





Asymptotic stability criterion:

$$\text{Re}\{\lambda_i(A)\} < 0 \quad \text{where}$$

$$A \triangleq \left.\frac{\partial f}{\partial x}\right|_{x = x^*}$$

Asymptotic stability criterion

$$|\lambda_i(A)| < 1 \quad \text{where } A \triangleq \left.\frac{\partial f}{\partial x}\right|_{x^*}$$

$|f'(x^*)| < 1$ for 1st order system

Triage criteria - -

Cobweb Diagrams