Lecture 7: Mathematical Background

Overview

- existence & uniqueness of ODEs
- Lipschitz continuity
- Normed linear spaces
- Fixed point theorems
- contraction mappings

Recap:

A k-dimensional manifold in $\mathbb{R}^n$ $(1 \leq k < n)$ is informally the solution to

$$\eta(x) = 0$$

$$\eta : \mathbb{R}^n \to \mathbb{R}^{n-k}$$

$z = h(y)$ is a center manifold $y \in \mathbb{R}^k$

$z = \mathbb{R}^{n-k}$

$$\omega(x) \triangleq z(x) - h(y(x)) = 0$$

examples

unit circle

$\{ x \in \mathbb{R}^2 \text{ s.t. } \eta(x) = x_1^2 + x_2^2 - 1 = 0 \}$ is 1-dim manifold in $\mathbb{R}^1$

unit sphere $(n-1)$ dim manifold

$$\{ x \in \mathbb{R}^n \text{ s.t. } \eta(x) \triangleq \sum_{i=1}^{n} x_i^2 - 1 = 0 \}$$

Mathematical Background

$$\dot{x} = f(x) \qquad x(0) = x_0$$

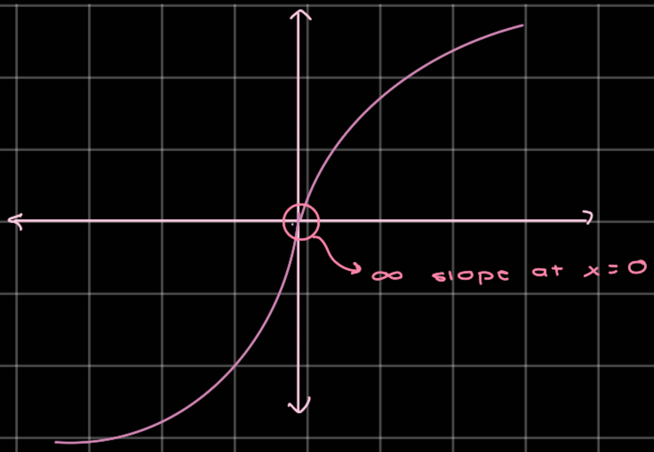
Q/ Do solns exist? Are they unique?

("existence & uniqueness")

A/ If $f(\cdot)$ is continuous (C^0) then a solution exists, but C^0 isn't sufficient for uniqueness

example: $\dot{x} = x^{\frac{1}{3}}$ with $x(0) = 0$

$x(t) = 0$, $x(t) = \left(\frac{2}{3} t\right)^{3/2}$ are both solutions



- sufficient condition for uniqueness "Lipschitz continuity"

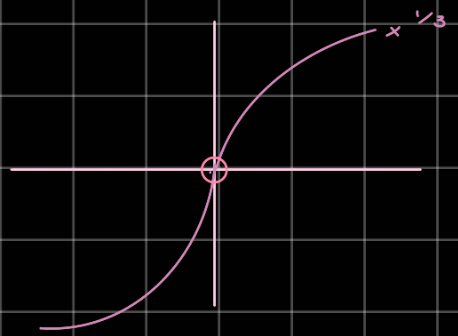
(more restrictive than C^0)

$$|f(x) - f(y)| \leq L|x-y| \quad (*)$$

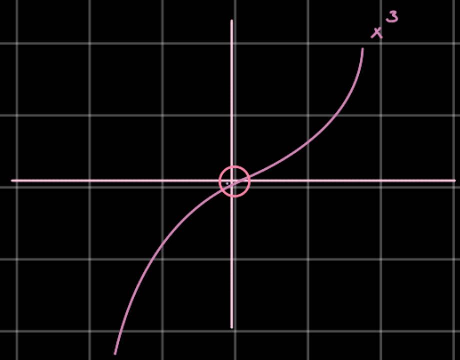
Uniqueness

- Def $f(\cdot)$ is locally Lipschitz if every point x_0 has a neighborhood where $(*)$ holds $\forall x, y$ in this neighborhood for some L

example



$(\cdot)^{1/3}$ isn't locally Lipschitz



$(\cdot)^3$ is locally Lipschitz:

$$x^3 - y^3 = \underbrace{(x^2 + xy + y^2)}_{\leq L}(x-y)$$

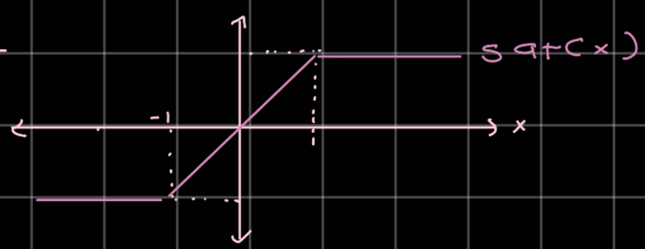
$$\Rightarrow |x^3 - y^3| \leq L|x-y|$$

- if $f(\cdot)$ is continuously differentiable (C^1), then it's locally Lipschitz (ex: x^3, x^2, e^x, etc)

→ the converse is not true:

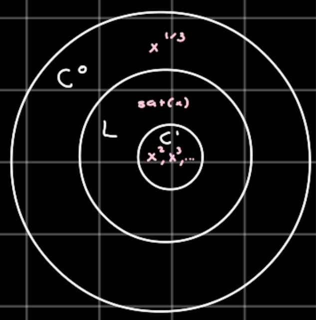
$$\text{local Lipschitz} \not\Rightarrow C^1$$

example



Not diffable at $x \pm 1$, but locally Lipschitz

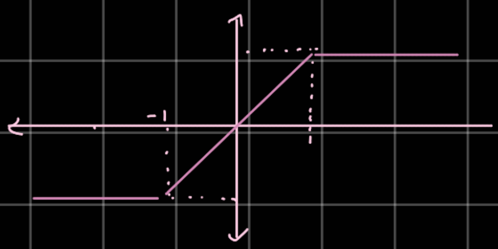
$$|sat(x) - sat(y)| \leq |x - y| \qquad (L = 1)$$



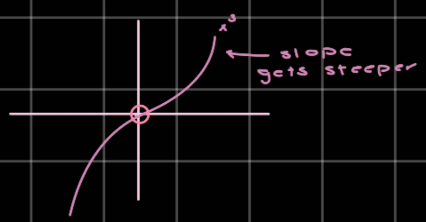
- Def: $f(\cdot)$ is globally Lipschitz if $(*)$ hold $\forall x, y \in \mathbb{R}^n$
  (ie, the same L works everywhere)

  example

  $sat(\cdot)$ is globally Lipschitz



$(\cdot)^3$ is not



- Suppose $f(\cdot)$ is C^1. Then, it's globally Lipschitz iff

  $\frac{\partial f}{\partial x}$ is bounded

$$L = \sup_x |f'(x)|$$

Existence

Preview of Existence Theorems:

1. $f(\cdot)$ is C^0 $\Rightarrow$ existence of solution $x(t)$ on finite interval $[0, t_f)$
2. $f(\cdot)$ locally Lipschitz $\Rightarrow$ existence & uniqueness on $[0, t_f)$
3. $f(\cdot)$ globally Lipschitz $\Rightarrow$ existence & uniqueness on $[0, \infty)$

examples:

- $\dot{x} = x^2$ (locally Lipschitz) admits unique solution on $[0, t_f)$ but $t_f < \infty$ (finite escape time)

- $\dot{x} = Ax$ globally Lipschitz $\Rightarrow$ no finite escape time

$$|Ax - Ay| \leq L|x-y| \quad \text{with} \quad L = \|A\|$$

Normed Linear Spaces

- def : $\mathcal{X}$ is a normed linear space if $\exists$ a real-valued norm satisfying
  1. $|x| \geq 0 \quad \forall x \in \mathcal{X}$, $|x| = 0$ iff $x = 0$
  2. $|x+y| \leq |x| + |y| \quad \forall x, y \in \mathcal{X}$ (triangle ineq.)
  3. $|\alpha x| = |\alpha| \cdot |x| \quad \forall \alpha \in \mathbb{R}$ & $x \in \mathcal{X}$
- def : A sequence $\{x_k\}$ in $\mathcal{X}$ is said to be a Cauchy sequence if

$$|x_k - x_m| \to 0 \quad \text{as} \quad k, m \to \infty$$

→ every convergent sequence is Cauchy.

The converse is not true

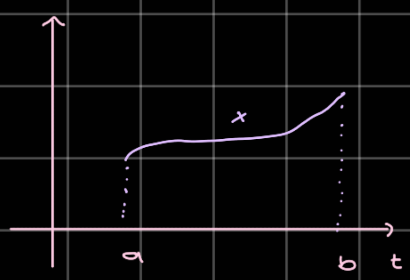
- def : $\mathcal{X}$ is a Banach space if every Cauchy sequence converges to an element in $\mathcal{X}$

→ All Euclidean spaces are Banach spaces

example

$C^n[a,b]$: the set of all continuous functions $[a,b] \to \mathbb{R}^n$ with norm:

$$|x|_c = \max_{t \in [a,b]} |x(t)|$$



1. $|x|_c \geq 0$ & $|x|_c = 0$ iff $x(t) \equiv 0$
2. $|x+y|_c = \max_{t \in [a,b]} |x(t) + y(t)| \leq \max_{t \in [a,b]} \{|x(t)| + |y(t)|\}$
   $\leq |x|_c + |y|_c$
3. $|\alpha \cdot x|_c = \max_{t \in [a,b]} |\alpha| \cdot |x(t)| = |\alpha| \cdot |x|_c$

It can be shown that $C^n[a,b]$ is a Banach space

Fixed Point Theorems (for discrete-time systems)

$$T(x) = x$$

Brouwer's Theorem (Euclidean spaces):

If U is a closed, bounded, convex subset of a Euclidean space & $T: U \to U$ is continuous, then T has a fixed point in U

Schauder's Theorem (Brouwer's Thm → Banach spaces)

If U is a closed, bounded subset of a Banach space X & $T: U \to U$ is $\underbrace{\text{completely continuous}}_{\text{continuous \& for any bounded set } B \subseteq U \text{ the closure of } T(B) \text{ is compact}}$

then T has a fixed point in U

Contraction Mapping Theorem:

If U is a closed subset of a Banach space & $T: U \to U$ is such that

$$|T(x) - T(y)| \leq \rho |x - y| \qquad \rho < 1 \quad \forall x, y \in U$$

then T has a unique fixed point in U & the solutions of $x_{n+1} = T(x_n)$ converge to