Lecture 8: Lyapunov Stability Theory

Overview:

- define Lyapunov stability notions
- Lyapunov Stability Theorems

Lyapunov Stability Theory

Consider a time invariant system

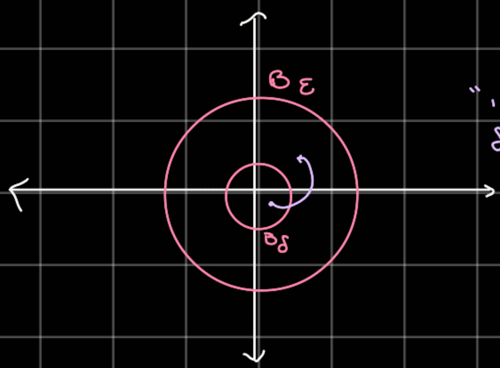
$$\dot{x} = F(x)$$

and assume equilibrium at $x = 0$; ie $F(0) = 0$. If the equilibrium of interest is $x^* \neq 0$, let $\tilde{x} = x - x^*$:

$$\dot{\tilde{x}} = F(x) = F(\tilde{x} + x^*) \triangleq \tilde{f}(\tilde{x}) \Rightarrow \tilde{f}(0) = 0$$

def: The equilibrium at $x = 0$ is stable, if for each $\varepsilon > 0$, $\exists\, \delta > 0$ s.t:

$$|x(0)| \leq \delta \Rightarrow |x(t)| \leq \varepsilon \quad \forall t \geq 0$$



"if you start within some δ of the origin, you must end w/in some epsilon"

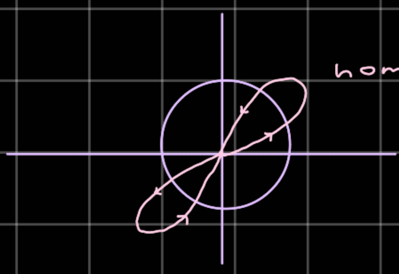
It is unstable otherwise.

def: asymptotically stable if you are stable & $x(t) \to 0$ $\forall\, x(0)$ in a neighborhood of $x = 0$

def: globally asymptotically stable if you're stable & $x(t) \to 0$ $\forall\, x(0)$.

Note: $x(t) \to 0$ doesn't necessarily imply stability

eg:



homoclinic orbit

Lyapunov's Stability Theorem

① Let D be an open, connected subset of $\mathbb{R}^n$ that includes $x = 0$. If $\exists$ C' funtion $V: D \to \mathbb{R}$ s.t:

$$V(0) = 0 \quad \& \quad \underbrace{V(x) > 0}_{\text{positive definite}} \quad \forall x \in D - \{0\}$$

then $x = 0$ is stable.

② If $\dot{V}(x) < 0 \quad \forall x \in D - \{0\}$

↳ neg. definite

then $x = 0$ is asymptotically stable

③ Additionally, $D = \mathbb{R}^n$ "if the domain is the entire space we're considering" &

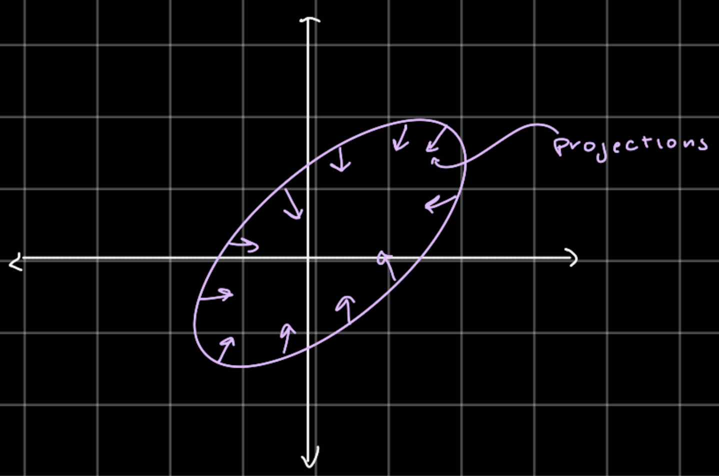
"the bound on our state goes to infinity" $|x| \to \infty \Rightarrow V(x) \to \infty$

radially unbounded

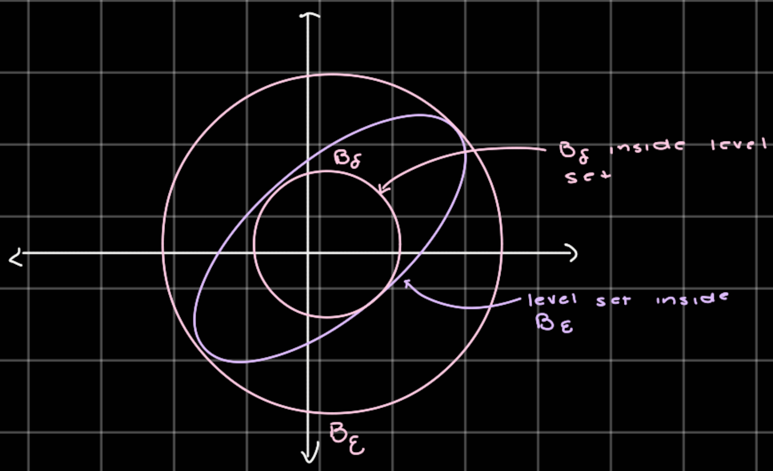
then $x = 0$ is globally asymptotically stable

Sketch of the Proof

The sets $\Omega_c \triangleq \{x : V(x) \leq c\}$ for constants c (upper bound) are called level sets of V and are positively invariant because $\nabla V(x)^T F(x) \leq 0$



Stability follows from this positive invariance:



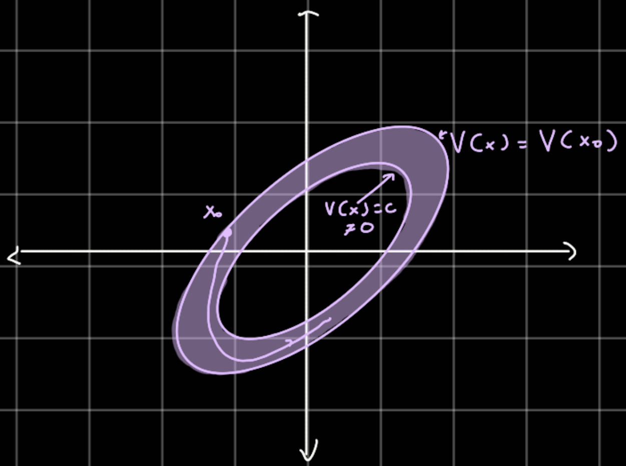
Asymptotic Stability:

Since $V(x(t))$ is decreasing & bounded below by 0, we conclude

$$V(x(t)) \longrightarrow c \geq 0$$

We'll show $c=0$ ($x(t) \to 0$) by contradiction.

Suppose $c \neq 0$:



Let $\gamma \triangleq \min\limits_{\{x : c \leq V(x) \leq V(x_0)\}} -\dot{V}(x) > 0$

where the maximum exists because it is evaluated over a bounded set & positive because $\dot{V}(x) < 0$ away from $x=0$.

Then

$$\dot{V}(x) \leq -\gamma \Rightarrow V(x(t)) \leq V(x_0) - \gamma t$$

$\Rightarrow V(x(t)) < 0$ for $t > \dfrac{V(x_0)}{\gamma}$ $\Rightarrow$ contradiction

because $V \geq 0 \Rightarrow c=0 \Rightarrow x(t) \to 0$

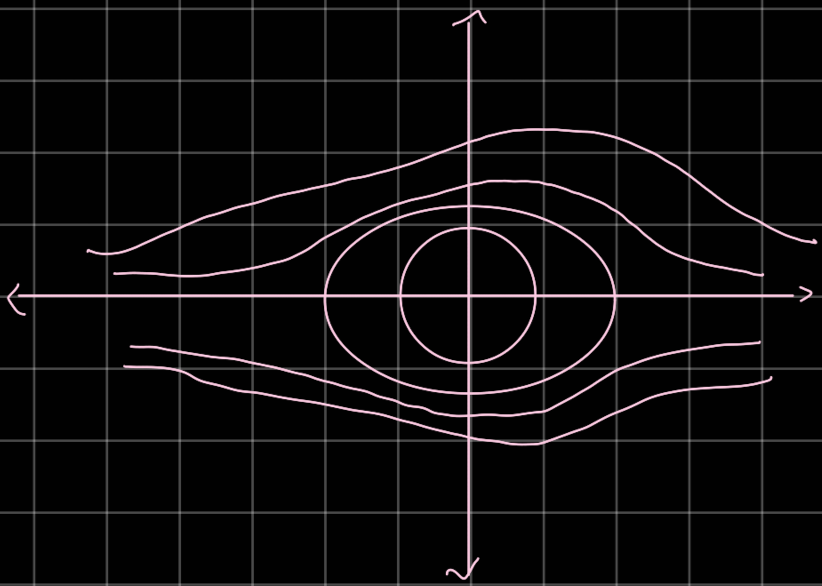
Global Asymptotic stability :

* Why do we need radial unboundedness?

ex:

$$V(x) = \frac{x_1^2}{1 + x_1^2} + x_2^2$$

set $x_2 = 0$, let $x_1 \to \infty$: $V(x) \to 1$ (not radially unbounded). Then Ω_c isn't a bounded set for $c \geq 1$:



$\Rightarrow$ $x_1(t)$ may grow unbounded while $V(x(t))$ is decreasing

Finding Lyapunov Functions

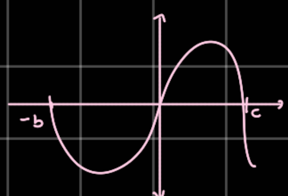
example:

$$\dot{x} = -g(x) \qquad x \in \mathbb{R}, \; x g(x) > 0 \quad \forall x \neq 0$$

$V(x) = \frac{1}{2} x^2$ is positive definite & radially unbounded. $\dot{V}(x) = -x g(x)$ is negative definite. Therefore $x = 0$ is globally asymptotically stable.

$\dot{V}(x) = -x g(x)$
$= \frac{\partial V}{\partial x} f(x)$
$= x F(x)$
$= \underbrace{-x g(x)}_{>0}$
$\dot{V}(x) < 0$
$\Rightarrow x = 0$ is globally asymptotically stable

If $x g(x) > 0$ only in $(-b, c) - \{0\}$, then take $D = (-b, c)$ $\Rightarrow$ $x = 0$ is locally asymptotically stable.



example :

$\dot{x}_1 = 2$

$\dot{x}_2 = -ax_2 - g(x_1) \qquad a \geq 0, \quad x g(x) > 0 \qquad \forall x \in (-b, c) - \{0\}$

$V(x) = \frac{1}{2} x_1^2 + \frac{1}{2} x_2^2$ doesn't work because $\dot{V}(x)$ $\underbrace{\text{is sign indefinite}}$:

$\dot{V}(x) > 0$ for some x

$\dot{V}(x) < 0$ for other x

→ instead consider

$$V(x) = \int_0^{x_1} g(y)\, dy + \frac{1}{2} x_2^2$$

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x)$$

$$= \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{bmatrix} f(x)$$

$$= \begin{bmatrix} g(x_1) & g(x_2) \end{bmatrix} f(x)$$

$$= g(x_1) x_2 - a x_2^2 - x_2 g(x_1)$$

$$= \underbrace{-a x_2^2}_{< 0} \Rightarrow \text{stable}$$

if $a > 0$

if $a = 0$, there's no asymptotic stability because $\dot{V}(x) = 0$

$\Rightarrow V(x(t)) = V(x(0))$

↳ Need LaSalle - Krasovskii invariance principle