Lecture 9: LaSalle-Krasovski Invariance Principle

Overview:

- LaSalle-Krasovski Invariance Principle

example from last lec (con't)

$\dot{x}_1 = x_2$ $\quad a \geq 0 \quad \forall x - \{0\}$

$\dot{x}_2 = -ax_2 - g(x_1) \quad xg(x) > 0$

$V(x) = \int_0^{x_1} g(y)\,dy + \frac{1}{2} x_2^2$

$\dot{V}(x) = \frac{\partial V}{\partial x} f(x)$ ← if < 0 asymptotically stable
if ≤ 0 stable

$= -ax_2^2 \leq 0$

↳ LaSalle's will help us reason about this because right now we can only claim stability

Theorem: LaSalle Invariance Principle

← this is general

Let $\Omega \subset D$ be a compact set that is positively invariant wrt the system $\dot{x} = F(x)$

Let $V: D \to \mathbb{R}$ be a continuously diff'able (C^1) st

$\dot{V}(x) \leq 0$ in Ω

Let E be the set of all points in Ω where

$\dot{V}(x) = 0$

Let M be the largest invariant set in E

If all of this is true, then every solution starting in Ω approaches M as $t \to \infty$

Corollary: LaSalle-Krasovski Invariance Principle

↳ specific

Let $x = 0$ be an equilibrium point for $\dot{x} = F(x)$

Let $V: D \to \mathbb{R}$ be a C^1 & positive definite on D, containing $x = 0$ s.t

$\dot{V}(x) \leq 0$ in D

Let $S = \{x \in D \text{ s.t. } \dot{V}(x) = 0\}$ and suppose no solution can stay identically in S, other than the trivial solution $x(t) \equiv 0$.

Then, the origin $x=0$ is asymptotically stable

Corollary: LaSalle's for Global Asymptotic Stability

Let $x=0$ be an equilibrium point

Let $V: \mathbb{R}^n \to \mathbb{R}$ be C^1 and radially unbounded & positive definite s.t.

$$\dot{V}(x) \leq 0 \quad \forall x \in \mathbb{R}^n$$

Let $S = \{x \in \mathbb{R}^n \text{ s.t. } \dot{V}(x) = 0\}$

$\Rightarrow$ Globally asymptotic stability

Note: D is often selected to be the level set

$$\Omega_c = \{x : V(x) \leq c\}$$

which is bounded s.t

$$\dot{V}(x) \leq 0 \text{ in } \Omega_c$$

Then we define $S = \{x \in \Omega_c : \dot{V}(x) = 0\}$ & let M be the largest invariant set in S.

Then $\forall x(0) \in \Omega_c$, $x(t) \to M$.

Note: If no soln other than $x(t) \equiv 0$ can stay identically in S then $M = \{0\}$ & we conclude asymptotic stability

example:

$$\dot{V}(x) = -a x_2^2$$

$$S = \{x \in \Omega_c \text{ s.t. } x_2 = 0\}$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -a x_2 - g(x_1) \end{cases} \qquad a > 0 \qquad \boxed{x\, g(x) > 0} \quad \forall x \neq 0$$

$x_2 \equiv 0 \Rightarrow \dot{x}_2 \equiv 0$

$\Rightarrow (-a x_2 - g(x_1)) \equiv 0$

$g(x_1) \equiv 0 \Rightarrow x_1(t) \equiv 0$

$\Rightarrow$ asymptotic stability

example:

$\dot{x}_1 = x_2$ $\qquad a > 0,\ b > 0$

$\dot{x}_2 = -a x_2 - b x_1$

$V(x) = \frac{b}{2} x_1^2 + \frac{1}{2} x_2^2$

$\Rightarrow \quad \dot{V}(x) = -a x_2^2$

$\Rightarrow$ asymptotic stability

Alternatively:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} x$$

Try: $V(x) = x^T P x$ for $P^T = P \succ 0$

$$\dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x$$

$$= x^T P A x + (x^T A^T) P x$$

$$= x^T \underbrace{(PA + A^T P)}_{= -Q \text{ for } Q = Q^T \succ 0} x$$

$$= \underbrace{-x^T Q x}_{< 0}$$

$\Rightarrow x = 0$ is asymptotically stable

Lyapunov's Equation:

$$A^T P + P A = -Q$$

Linear Systems

$\dot{x} = Ax, \qquad x \in \mathbb{R}^n$

$x = 0$ is eq. pt.

$x = 0$ iff $\mathrm{Re}\{\lambda_i\} \leq 0 \quad \forall i = 1, \dots, n$ and

λ_i on the imaginaries have Jordan

blocks of order one.

$\Leftrightarrow$ if λ_i has a multiplicity q then

$\mathrm{rank}\{\lambda I - A\} = n - q$

example :

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \lambda_{1,2} = 0$$

$$\operatorname{rank}(\lambda I - A) = 1 \neq 2 - 2 = 0$$

$\Rightarrow$ unstable

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \lambda_{1,2} = 0$$

$$\operatorname{rk}(\lambda I - A) = 0$$

$\Rightarrow$ stable

A is "Horwitz" when $\operatorname{Re}\{\lambda_i\} < 0 \quad \forall \lambda_i$

$\Rightarrow$ asymptotic stability

$x = 0$ is asymp. stable IFF A is Horwitz

Lyapunov for Linear Systems

$$V(x) = x^T P x \qquad P = P^T \succ 0$$

$$\dot{V}(x) = x^T (A^T P + PA) x$$

if $\exists P = P^T \succ 0$ s.t.

$$A^T P + PA = -Q \prec 0$$

Then A is Horwitz & the converse is true

Theorem: A is Horwitz iff for any $Q = Q^T \succ 0$ $\exists P = P^T \succ 0$ such that $A^T P + PA = -Q$ & the solution P is unique

Invariance Principle Applied to Linear Systems

- We can relax

$$A^T P + PA = -Q \preceq 0$$

A is Horowitz iff Q is semidefinite

Pf sketch :

Decompose $Q = C^T C \qquad C \in \mathbb{R}^{r \times n} \qquad r = \operatorname{rk}(Q)$

$$\dot{V}(x) = -x^T Q x = -x^T C^T \underbrace{C x}_{y} = -y^T y$$

$y \doteq Cx$

The invariance principle guarantees asymptotic stability if

$$y(t) = Cx(t) \equiv 0 \rightarrow x(t) \equiv 0$$

This is true if (C, A) is observable

$$\theta = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \qquad rk(\theta) = n$$

$$y(t) \equiv 0 \Rightarrow x(t) \equiv 0$$

example :

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}}_{A} x$$

select $Q = \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix}$

$$\Rightarrow Q \succcurlyeq 0$$

$Q = C^T C$ where $C = [0 \quad \sqrt{a}]$

$$\theta = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{a} \\ -b\sqrt{a} & -a\sqrt{a} \end{bmatrix}$$

$\Rightarrow rk(\theta) = 2$ if $b \neq 0$

$\Rightarrow$ asymptotically stable if $b \neq 0$

1. Pick Q
2. Solve for P $\quad A^T P + PA = -Q$
3. $V(x) = x^T P x$
4. For $\dot{x} = Ax$, we know $\dot{V}(x) = -x^T Q x < 0$

Ways to solve $A^T P + PA = -Q$

Matlab: lyap $(A', Q) \Rightarrow P$

Let $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}$ Take $Q = I$

Let $A = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$

$$A^T P + PA = -Q$$

$$\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = -I$$

$$\begin{bmatrix} P_{12} & P_{22} \\ -P_{11} - P_{12} & -P_{12} - P_{22} \end{bmatrix} + \begin{bmatrix} P_{12} & -P_{11} - P_{12} \\ P_{22} & -P_{12} - P_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 2P_{12} & -P_{11} - P_{12} + P_{22} \\ -P_{11} - P_{12} - P_{222} & -2P_{12} - P_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & 0 \\ & & \\ & & \end{bmatrix}\begin{bmatrix} P_{11} \\ P_{12} \\ \end{bmatrix}$$